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Solvability of a Boundary Value Problem with the Nonlinearity Satisfying a Sign Condition

CHAITAN P. GUPTA

Department of Mathematical Sciences, Northern Illinois University, DeKalb, Illinois 60115

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1. INTRODUCTION

Let f: $[0, \pi] \times \mathbb{R} \to \mathbb{R}$ be a given function satisfying Caratheodory's conditions and h: $[0, \pi] \to \mathbb{R}$ be a given function in $L^1[0, \pi]$. The boundary value problem

$$\begin{gathered} u''(x) + u(x) + f(x, u(x)) = h(x), \qquad x \in [0, \pi] \\ u(0) = u(\pi) = 0 \end{gathered} \tag{1.1}$$

has been recently studied by Mawhin, Ward and Willem [9] when $f(x, \cdot)$ is nondecreasing for each $x \in [0, \pi]$, $h(x) \equiv 0$ under a condition on $F(x, u) = \int_0^u f(x, v)\, dv$ and they give a necessary and sufficient condition for the solvability of (1.1). Problem (1.1) was studied by Fucik [3], Schechter, Shapiro, and Snow [10], Cesari and Kannan [2], and Ahmad [1], who considered the existence of a solution of (1.1) when $f(x, u) = g(u) - h(x)$ with $h \in L^2[0, \pi]$, g is continuous nondecreasing, and

$$|g(u)| \leqslant C_1 + C_2\,|u|, \qquad u \in \mathbb{R} \tag{1.2}$$

for some $C_1 \geqslant 0$ and $C_2 > 0$. In [3], the existence of a solution of (1.1) is obtained when $h(x) \equiv 0$, under the supplementary conditions that

$$\begin{gathered} g(u) \text{ is nondecreasing on } \mathbb{R} \\ g(-u) = -g(u), \qquad \lim_{u \to \infty} g(u) = \infty \end{gathered} \tag{1.3}$$

and

$$C_2 < 0.0962.$$

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These supplementary conditions were weakened by Cesari and Kannan [2] to

$$\limsup_{|u| \to \infty} \frac{g(u)}{u} = C_2 \tag{1.4}$$

and

$$C_2 < 0.443. \tag{1.5}$$

Noticing that the first two eigenvalues of the linear eigenvalue problem $-u'' - u = \lambda u$, $u(0) = u(\pi) = 0$ are $\lambda_1 = 0$ and $\lambda_2 = 3$ with $\lambda_2 - \lambda_1 = 3$, Ahmad [1] obtained the existence of a solution for (1.1) when $h \in L^2[0, \pi]$ with

$$g^*(-\infty) \int_0^\pi \sin x \, dx < \int_0^\pi h(x) \sin x \, dx < g^*(\infty) \int_0^\pi \sin x \, dx, \tag{1.6}$$

where $g^*(-\infty) = \limsup_{u \to -\infty} g(u)$, $g^*(\infty) = \liminf_{u \to +\infty} g(u)$ and g satisfied (1.2) with

$$0 < C_2 < 3. \tag{1.7}$$

The condition on $F(x, u) = \int_0^u f(x, v) \, dv$ imposed by Mawhin *et al.* in [9] is implied by (1.2), (1.7) in the special case $f(x, u) = g(u) - h(x)$. We observe that a necessary condition for the solvability of (1.1), in view of the Fredholm alternative, is

$$\int_0^\pi h(x) \sin x \, dx = 0. \tag{1.8}$$

Now, the necessary and sufficient condition on $f(x, u)$ when $h(x) \equiv 0$ given by Mawhin *et al.* [9] reduces to (1.8) in the special case of $f(x, u) = g(u) - h(x)$, $h(x) \in L^2[0, \pi]$.

The purpose of this paper is to obtain the existence of a solution for (1.1) when $h(x) \in L^1[0, \pi]$ satisfying (1.8) and $f(x, u)$ satisfies the supplementary conditions

$$f(x, u)\, u \geqslant 0 \qquad \text{for } u \in \mathbb{R} \tag{1.9}$$

and there is a constant $\beta \geqslant 0$ such that

$$\limsup_{|u| \to \infty} \frac{f(x, u)}{u} = \beta < 3 \tag{1.10}$$

uniformly for $x \in [0, \pi]$.

We also study the boundary value problem

$$\begin{aligned} -u'' - u + f(x, u) &= h(x) \\ u(0) = u(\pi) &= 0 \end{aligned} \tag{1.11}$$

and obtain the existence of a solution of (1.11) under (1.8) and (1.9).

Our method uses the version of the Leray–Schauder continuation theorem as given by Mawhin in [6–8] and is different from the method of Mawhin *et al.* [9], which is variational, and which makes essential use of the assumption that $f(x, \cdot)$ is nondecreasing for each x in $[0, \pi]$. The estimates used here are similar to those in [4].

2. Main Results

Let X, Y denote the Banach spaces $X = C[0, \pi]$ and $Y = L^1[0, \pi]$ with their usual norms and let H denote the Hilbert space $L^2[0, \pi]$. Let Y_2 be the subspace of Y spanned by the function $\sin x$, i.e.,

$$Y_2 = \{u \in Y \mid u(x) = \alpha \sin x \text{ a.e. for some } \alpha \in \mathbb{R}\}, \tag{2.1}$$

and let Y_1 be the subspace of Y such that $Y = Y_1 \oplus Y_2$. We note that for $u \in Y$ we can write

$$u(x) = u(x) - \left(\frac{2}{\pi}\int_0^\pi u(t) \sin t \, dt\right) \sin x + \left(\frac{2}{\pi}\int_0^\pi u(t) \sin t \, dt\right) \sin x, \tag{2.2}$$

$x \in [0, \pi]$. We define the canonical projection operators $P\colon Y \to Y_1$, $Q\colon Y \to Y_2$ by

$$\begin{aligned} P(u) &= u(x) - \left(\frac{2}{\pi}\int_0^\pi u(t) \sin t \, dt\right) \sin x \\ Q(u) &= \left(\frac{2}{\pi}\int_0^\pi u(t) \sin t \, dt\right) \sin x \end{aligned} \tag{2.3}$$

for $u \in Y$. Clearly, $Q = I - P$, where I denotes the identity mapping on Y, and the projections P and Q are continuous. Now let $X_2 = X \cap Y_2$. Clearly X_2 is a closed subspace of X. Let X_1 be the closed subspace of X such that $X = X_1 \oplus X_2$. We note that $P(X) \subset X_1$, $Q(X) \subset X_2$ and the projections $P \mid X\colon X \to X_1$ and $Q \mid X\colon X \to X_2$ are continuous. Similarly, we obtain $H = H_1 \oplus H_2$ and the continuous projections $P \mid H\colon H \to H_1$, $Q \mid H\colon H \to H_2$. In the following, X, Y, H, P, Q, etc., will refer to the Banach spaces, Hilbert space, and the projections as defined above and we shall not distinguish between P, $P \mid X$, $P \mid H$ (resp. Q, $Q \mid X$, $Q \mid H$) and depend on the context for the proper meaning.

Also for $u \in X$, $v \in Y$ let $(u, v) = \int_0^\pi u(x)\, v(x)\, dx$ denote the duality pairing between X and Y. We note that for $u \in X$, $v \in Y$ so that $u = Pu + Qu$, $v = Pv + Qv$, we have

$$(u, v) = (Pu, Pv) + (Qu, Qv). \tag{2.4}$$

Define a linear operator $L\colon D(L) \subset X \to Y$ by setting

$$D(L) = \{u \in X \mid u'(t) \in AC[0, \pi],\ u(0) = u(\pi) = 0\} \tag{2.5}$$

and for $u \in D(L)$

$$Lu = -u'' - u. \tag{2.6}$$

(Here $AC[0, \pi]$ denotes the space of real valued absolutely continuous functions on $[0, \pi]$.) Now, for $u \in D(L)$ we have

$$(Lu, u) = -\int_0^\pi u''u - \int_0^\pi u^2 = \int_0^\pi (u')^2 - \int_0^\pi u^2\, dx \geqslant 0 \tag{2.7}$$

in view of Wirtinger's inquality.

Let now, for $h \in Y_1$, i.e., $h \in L^1(0, \pi)$ such that $\int_0^\pi h(x) \sin x\, dx = 0$, Kh denote the unique solution of the problem

$$-u''(x) - u(x) = h(x)$$
$$u(0) = u(\pi) = 0$$

such that $\int_0^\pi u(x) \sin x\, dx = 0$. It is immediate that $K\colon Y_1 \to X_1$ is a bounded linear mapping such that for

$$\begin{gathered} u \in Y,\ KP(u) \in D(L), \\ LKP(u) = P(u) \qquad \text{and} \qquad (KP(u), P(u)) \geqslant 0. \end{gathered} \tag{2.8}$$

Also we see, using Fourier series and Parseval inequality, for $u \in H_1$ (i.e., $u \in L^2[0, \pi]$ with $\int_0^\pi u(x) \sin x\, dx = 0$), that

$$(Ku, u) \leqslant \tfrac{1}{3} \|u\|_H^2 \tag{2.9}$$

with equality if and only if u has the form

$$u(x) = \alpha \sin 2x$$

for some $\alpha \in \mathbb{R}$.

DEFINITION 2.1. $f\colon [0, \pi] \times \mathbb{R} \times \mathbb{R}$ satisfies Caratheodory's conditions for $L^1[0, \pi]$ (resp. $L^2[0, \pi]$) if $f(x, \cdot)$ is continuous for a.e. $x \in [0, \pi]$,

$f(\cdot, u)$ is measurable on $[0, \pi]$ for each $u \in \mathbb{R}$, and for each $r \in \mathbb{R}$ there is a function $\alpha_r(x) \in L^1[0, \pi]$ (resp. $L^2[0, \pi]$) such that $|f(x, u)| \leqslant \alpha_r(x)$ whenever $|u| \leqslant r$.

Let f: $[0, \pi] \times \mathbb{R} \to \mathbb{R}$ be given and N: $X \to Y$ be the nonlinear mapping defined by

$$(Nu)(x) = f(x, u(x)), \qquad x \in [0, \pi] \tag{2.10}$$

for $u \in X$.

For $h(x) \in Y = L^1[0, \pi]$ with $\int_0^\pi h(x) \sin x \, dx = 0$, the boundary value problem

$$\begin{gathered} -u'' - u + f(x,u) = h(x), \qquad x \in [0, \pi] \\ u(0) = u(\pi) = 0 \end{gathered} \tag{2.11}$$

now reduces to the functional equation

$$Lu + Nu = h \tag{2.12}$$

in X with a given $h \in Y_1$.

THEOREM 2.2. *Let f: $[0, \pi] \times \mathbb{R} \to \mathbb{R}$ satisfy Caratheodory's conditions for $L^1[0, \pi]$ and*

$$f(x, u)\, u \geqslant 0 \qquad \textit{for} \quad x \in [0, \pi],\ u \in \mathbb{R}.$$

Then, for each $h \in Y = L^1[0, \pi]$ with $\int_0^\pi h(t) \sin t \, dt = 0$, the boundary value problem

$$\begin{gathered} -u'' - u + f(x, u) = h(x), \qquad x \text{ in } [0, \pi] \\ u(0) = u(\pi) = 0 \end{gathered} \tag{2.13}$$

has at least one solution u in $X = C[0, \pi]$.

Proof. In the following X, Y, X_1, X_2, Y_1, Y_2, L, K, P, Q, N will be as defined above from the beginning of this section to just before the statement of Theorem 2.2.

As noted above, the boundary value problem (2.13) reduces to the functional equation

$$Lu + Nu = h \tag{2.14}$$

in X with $h \in Y_1$. Now to solve the functional equation (2.14) it suffices to solve the system of equations

$$\begin{gathered} Pu + KPNu = h_1, \\ QNu = 0, \end{gathered} \tag{2.15}$$

$u \in X$, $h_1 = Kh$ (note that since $h \in Y_1$, $Ph = h$, $Qh = 0$). Indeed, if $u \in X$ is a solution of (2.15) then $u \in D(L)$ and

$$LPu + LKPNu = Lu + PNu = Lh_1 = h,$$

$$QNu = 0,$$

which gives on adding that $Lu + Nu = h$.

Now, (2.15) is clearly equivalent to the single equation

$$Pu + QNu + KPNu = h_1, \tag{2.16}$$

which has the form of a compact perturbation of the Fredholm operator P of index zero. We can therefore apply the version given in [6, Theorem 1 and Corollary 1], [7, Theorem IV.4], or [8] of the Leray–Schauder continuation theorem, which ensures the existence of a solution for (2.16) if the set of solutions of the family of equations

$$Pu + (1-\lambda)\, Qu + \lambda QNu + \lambda KPNu = \lambda h_1, \tag{2.17}$$

$\lambda \in]0, 1[$, is a priori bounded independently of λ. Notice that (2.17) is then equivalent to the system of equations

$$\begin{aligned} Pu + \lambda KPNu &= \lambda h_1 \\ (1-\lambda)\, Qu + \lambda QNu &= 0. \end{aligned} \tag{2.18}$$

If $u_\lambda \in X$ is a solution of (2.18) for $\lambda \in]0, 1[$ then $u_\lambda \in D(L)$ and

$$(Pu_\lambda, PNu_\lambda) + \lambda(KPNu_\lambda, PNu_\lambda) = \lambda(h_1, PNu_\lambda)$$

$$(1-\lambda)(Qu_\lambda, QNu_\lambda) + \lambda(QNu_\lambda, QNu_\lambda) = 0.$$

Consequently, we have using (2.8) that

$$\begin{aligned} (Pu_\lambda, PNu_\lambda) &\leqslant \lambda(h_1, PNu_\lambda) \\ (Qu_\lambda, QNu_\lambda) &\leqslant 0. \end{aligned} \tag{2.19}$$

Next it is easy to prove from our assumptions on f that for every $k \in \mathbb{R}$, $k \geqslant 0$, there is a constant $C(k) \geqslant 0$, such that

$$(Nu, u) \geqslant k\, \|Nu\|_Y - C(k), \qquad u \in Y. \tag{2.20}$$

Using now (2.20) and (2.19) we see that for each $k \in \mathbb{R}$, $k \geqslant 0$, there is a constant $C(k) \geqslant 0$, such that

$$\begin{aligned} k\, \|Nu_\lambda\|_Y - C(k) \leqslant (Nu_\lambda, u_\lambda) &\leqslant \lambda(h_1, PNu_\lambda) \\ &\leqslant \|h_1\|_X\, \|PNu_\lambda\|_Y \\ &\leqslant C_0\, \|h_1\|_X\, \|Nu_\lambda\|_Y, \end{aligned}$$

where $C_0 \geqslant 0$ is such that $\|Pu\|_Y \leqslant C_0\, \|u\|_Y$.

Thus,

$$(k - C_0 \|h_1\|_X) \|Nu_\lambda\|_Y \leqslant C(k). \tag{2.21}$$

Also we obtain from the first equation in (2.18) that

$$\begin{aligned} \|Pu_\lambda\|_X &\leqslant \|KPN_\lambda\|_X + \|h_1\|_X \\ &\leqslant \|K\| \ \|PNu_\lambda\|_Y + \|h_1\|_X \\ &\leqslant C_0 \ \|K\| \ \|Nu_\lambda\|_Y + \|h_1\|_X. \end{aligned} \tag{2.22}$$

Taking $k > C_0 \|h_1\|_X$, we see from (2.21) and (2.22) that there is a constant $C \geqslant 0$ independent of λ in $]0, 1[$ such that

$$\|Nu_\lambda\|_Y \leqslant C, \qquad \|Pu_\lambda\|_X \leqslant C, \qquad \lambda \in]0, 1[, \tag{2.23}$$

It only remains to prove that there is a constant C_1, independent of $\lambda \in]0, 1[$ such that $\|Qu_\lambda\|_X \leqslant C_1$, $\lambda \in]0, 1[$. Let us suppose, on the other hand, that the set

$$\{\|Qu_\lambda\|_X : \lambda \in]0, 1[\} \text{ is unbounded.} \tag{2.24}$$

We now have from the first equation in (2.18) that

$$LPu_\lambda + \lambda LKPNu_\lambda = \lambda Lh_1,$$

i.e.,

$$LPu_\lambda + \lambda PNu_\lambda = \lambda h$$

so that

$$\begin{aligned} \|LPu_\lambda\|_Y &\leqslant \lambda \ \|PNu_\lambda\|_Y + \lambda \ \|h\|_Y \\ &\leqslant \|PNu_\lambda\|_Y + \|h\|_Y \\ &\leqslant C_0 \ \|Nu_\lambda\|_Y + \|h\|_Y \leqslant C_1 \end{aligned}$$

for $\lambda \in]0, 1[$, where $C_1 = C_0 C + \|h\|_Y$.

Since now

$$LPu_\lambda = -(Pu_\lambda)'' - Pu_\lambda,$$

$\|Pu_\lambda\|_X \leqslant C$, and $u_\lambda(0) = u_\lambda(\pi) = 0$, we see easily that $\|(Pu_\lambda)''\|_Y$ is bounded independently of λ and accordingly there is a constant $C_2 > 0$, independent of $\lambda \in]0, 1[$ such that

$$\|(Pu_\lambda)'\|_X \leqslant C_2.$$

We next use the well-known estimate (see, e.g., estimate (16) of [9])

$$\left|\frac{v(x)}{\sin x}\right| \leqslant \frac{\pi}{2} \max_{s \in [0, \pi]} |v'(s)|$$

for $v \in X$, $v(0) = v(\pi) = 0$ to get

$$|(Pu_\lambda)(x)| \leqslant \frac{\pi}{2} C_2 \sin x \qquad \text{for } x \in [0, \pi],\ \lambda \in\ ]0, 1[. \tag{2.25}$$

Now by (2.24) we see that there is a sequence $\{\lambda_n\}$, $\lambda_n \in\ ]0, 1[$, such that

$$\|Qu_{\lambda_n}\|_X = \left|\frac{2}{\pi} \int_0^\pi u_{\lambda_n}(t) \sin t\, dt\right| \to \infty$$

as $n \to \infty$. We may now assume that

$$\int_0^\pi u_{\lambda_n}(t) \sin t \to \infty \tag{2.26}$$

as $n \to \infty$, so that there is an n_0 such that

$$\int_0^\pi u_{\lambda_n}(t) \sin t\, dt \geqslant \frac{\pi^2}{4} C_2 \qquad \text{for } n \geqslant n_0. \tag{2.27}$$

So, for $n \geqslant n_0$, $x \in [0, \pi]$ we have, using (2.25), (2.27), that

$$\begin{aligned} u_{\lambda_n}(x) &= Qu_{\lambda_n}(x) + Pu_{\lambda_n}(x) \\ &= \frac{2}{\pi}\left(\int_0^\pi u_{\lambda_n}(t) \sin t\, dt\right) \sin x + Pu_{\lambda_n}(x) \\ &\geqslant \frac{2}{\pi} \cdot \frac{\pi^2}{4} C_2 \sin x - \frac{\pi}{2} C_2 \sin x \geqslant 0. \end{aligned}$$

Since now $f(x, v)\, v \geqslant 0$ for $x \in [0, \pi]$, $v \in \mathbb{R}$, we have $f(x, u_{\lambda_n}(x)) \geqslant 0$ for $n \geqslant n_0$, $x \in [0, \pi]$ and

$$(QNu_{\lambda_n}, Qu_{\lambda_n}) \geqslant 0 \qquad \text{for} \quad n \geqslant n_0.$$

It then follows from the second equation in (2.18) that

$$(1 - \lambda_n)(Qu_{\lambda_n}, Qu_{\lambda_n}) = (1 - \lambda_n) \cdot \frac{2}{\pi}\left(\int_0^\pi u_{\lambda_n}(t) \sin t\, dt\right)^2 \leqslant 0$$

for $n \geqslant n_0$, a contradiction. Similarly, assuming $\int_0^\pi u_{\lambda_n}(t)\ \sin t\, dt \to -\infty$ leads to a contradiction. Thus the set $\{\|Qu_\lambda\|_X \colon \lambda \in\ ]0, 1[\}$ is bounded by a constant independent of $\lambda \in]0, 1[$. We have, accordingly, proved that the set of solutions of the system of equations (2.18) is bounded independently of $\lambda \in]0, 1[$ and the proof of the theorem is complete.

Remark 2.3. It was remarked by Mawhin *et al.* in [9] that solvability of (2.13) can be obtained by a simple upper and lower solution argument like in Kazdan and Warner [5] when $f(x, \cdot)$ is nondecreasing for each x in $[0, \pi]$ and $h(x) \equiv 0$. The upper and lower solution argument makes essential use of the assumption that $f(x, \cdot)$ is nondecreasing and does not apply to the situation of Theorem 2.2.

THEOREM 2.4. *Let $f\colon [0, \pi] \times \mathbb{R} \to \mathbb{R}$ satisfy Caratheodory's conditions for $L^2[0, \pi]$ and*

(i) $f(x, u)\, u \geqslant 0$ *for* $x \in [0, \pi]$, $u \in \mathbb{R}$,

(ii) *there is a constant $\beta \geqslant 0$ such that*

$$\limsup_{|u| \to \infty} \frac{f(x, u)}{u} = \beta < 3$$

uniformly for $x \in [0, \pi]$.

Then, for each $h \in Y = L^1[0, \pi]$ with $\int_0^\pi h(t) \sin t\, dt = 0$, the boundary value problem

$$\begin{gathered} u'' + u + f(x, u) = h(x), \qquad x \text{ in } [0, \pi] \\ u(0) = u(\pi) = 0 \end{gathered} \tag{2.28}$$

has at least one solution u in $X = C[0, \pi]$.

Proof. Let us set $\tilde{L} = -L$ and $\tilde{K} = -K$, where $L\colon D(L) \subset X \to Y$ is the linear operator defined by (2.5), (2.6) and $K\colon Y_1 \to X_1$ is the bounded linear mapping as in (2.8), (2.9). Accordingly, we have for

$$u \in Y,\ \tilde{K}P(u) \in D(\tilde{L}), \qquad \tilde{L}\tilde{K}P(u) = P(u), \tag{2.29}$$

and for $u \in H_1$,

$$(\tilde{K}u, u) \geqslant -\tfrac{1}{3} \|u\|_H^2. \tag{2.30}$$

Now, as in the proof of Theorem 2.2, the boundary value problem (2.28) reduces to

$$\tilde{L}u + Nu = h$$

in X with $h \in Y_1$ and it suffices to show that the set of solutions of the system of equations

$$\begin{gathered} Pu + \lambda\tilde{K}PNu = \lambda h_1, \\ (1 - \lambda)\, Qu + \lambda QNu = 0, \end{gathered} \tag{2.31}$$

where $h_1 = \tilde{K}h$, is a priori bounded in X independently of $\lambda \in]0, 1[$.

Let now $u_\lambda \in X$ be a solution of (2.31) for $\lambda \in \,]0, 1[$. Then, we obtain, as in Theorem 2.1,

$$(Pu_\lambda, PNu_\lambda) + \lambda(KPNu_\lambda, PNu_\lambda) = \lambda(h_1, PNu_\lambda)$$
$$(1-\lambda)(Qu_\lambda, QNu_\lambda) + \lambda(QNu_\lambda, QNu_\lambda) = 0$$

and, hence, we get using (2.30)

$$\begin{aligned}(Pu_\lambda, PNu_\lambda) - \tfrac{1}{3}\,\|PNu_\lambda\|_H^2 &\leqslant \lambda(h_1, PNu_\lambda)\\ (Qu_\lambda, QNu_\lambda) &\leqslant 0.\end{aligned} \tag{2.32}$$

Since now $\limsup_{|u| \to \infty}(f(x, u)/u) = \beta < 3$ uniformly in $x \in [0, \pi]$, we see, choosing $\varepsilon > 0$ such that $\beta + \varepsilon < 3$, that there is a constant $C(\varepsilon) > 0$ such that

$$(Nu, u) \geqslant \frac{1}{\beta + \varepsilon}\,\|Nu\|_H^2 - C(\varepsilon) \tag{2.33}$$

for $u \in H$. We next have from (2.32), (2.33) that

$$\begin{aligned}\frac{1}{\beta+\varepsilon}\,\|Nu_\lambda\|_H^2 - C(\varepsilon) \leqslant (Nu_\lambda, u_\lambda) &\leqslant \frac{1}{3}\,\|PNu_\lambda\|_H^2 + \|h_1\|_X\,\|PNu_\lambda\|_Y\\ &\leqslant \frac{1}{3}\,\|Nu_\lambda\|_H^2 + C_0\,\|h_1\|_X\,\|Nu_\lambda\|_Y.\end{aligned}$$

Consequently,

$$\left(\frac{1}{\beta+\varepsilon} - \frac{1}{3}\right)\|Nu_\lambda\|_H^2 \leqslant \sqrt{\pi}\,C_0\,\|h_1\|_X\,\|Nu_\lambda\|_H + C(\varepsilon)$$

so that there is a constant $C > 0$, independent of $\lambda \in \,]0, 1[$, such that

$$\|Nu_\lambda\|_H \leqslant C. \tag{2.34}$$

Also we obtain from the first equation in (2.31) that

$$\begin{aligned}\|Pu_\lambda\|_X &\leqslant \|\tilde{K}PNu_\lambda\|_X + \|h_1\|_X\\ &\leqslant \|\tilde{K}\|\,\|PNu_\lambda\|_Y + \|h_1\|_X\\ &\leqslant \sqrt{\pi}\,C_0\,\|\tilde{K}\|\,\|Nu_\lambda\|_H + \|h_1\|_X\\ &\leqslant \sqrt{\pi}\,C_0\,\|\tilde{K}\|\,C + \|h_1\|_X \equiv C_1.\end{aligned}$$

The boundedness of $\{\|Qu_\lambda\|_X : \lambda \in \,]0, 1[\}$ now follows as in the proof of Theorem 2.1 above.

Thus, we have shown that the set of solutions of (2.31) is, a priori, bounded in X independently of $\lambda \in]0, 1[$ and the proof of the theorem is complete.

Remark 2.5. As an example we mention the boundary value problem

$$u'' + u + \frac{u}{1+u^2} = \cos x, \qquad x \in [0, \pi]$$

$$u(0) = u(\pi) = 0.$$

It is clear that the results of [1–3, 7, 10] do not apply to this example, since $g(u) = u/(1+u^2)$ is not nondecreasing and $g^*(-\infty) = g^*(\infty) = 0$. On the other hand, existence of a solution for (2.35) is immediate from Theorem 2.4.

Remark 2.6. The proofs of Theorems 2.2 and 2.4 can be easily adapted to the case when $f(x, \cdot)$ is nondecreasing for each x in $[0, \pi]$ and satisfies the necessary and sufficient condition imposed in [9]. In this way our theorems provide a nonvariational proof for the results of [9].

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